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Polynomial projectors preserving homogeneous partial differential equations

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Abstract

A polynomial projector Π of degree d on $H(\mathbb{C}^n)$ is said to *preserve homogeneous partial differential equations* (*HPDE*) of degree k if for every $f \in H(\mathbb{C}^n)$ and every homogeneous polynomial of degree k, $q(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$, there holds the implication: $q(D)f = 0 \Rightarrow q(D)\Pi(f) = 0$. We prove that a polynomial projector Π preserves HPDE of degree k, $1 \leqslant k \leqslant d$, if and only if there are analytic functionals $\mu_k, \mu_{k+1}, \ldots, \mu_d \in H'(\mathbb{C}^n)$ with $\mu_i(1) \neq 0$, $i = k, \ldots, d$, such that Π is represented in the following form

$$\Pi(f) = \sum_{|\alpha|<k} a_\alpha(f) u_\alpha + \sum_{k \leqslant |\alpha| \leqslant d} D^\alpha \mu_{|\alpha|} u_\alpha$$

with some a_α's $\in H'(\mathbb{C}^n)$, $|\alpha| < k$, where $u_\alpha(z) := z^\alpha/\alpha!$. Moreover, we give an example of polynomial projectors preserving HPDE of degree k $(k \geqslant 1)$ without preserving HPDE of smaller degree. We also give a characterization of Abel–Gontcharoff projectors as the only Birkhoff polynomial projectors that preserve all HPDE.

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Keywords: Polynomial projector preserving homogeneous partial differential equations; Space of interpolation conditions; D-Taylor projector; Birkhoff projector; Abel–Gontcharoff projector

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1. Introduction

As usual, we denote by $H(\mathbb{C}^n)$ the space of entire functions on $\mathbb{C}^n$ equipped with its usual compact convergence topology, and $\mathcal{P}_d(\mathbb{C}^n)$ the space of polynomials on $\mathbb{C}^n$ of total degree at most d. A *polynomial projector of degree d* is defined as a continuous linear map Π from $H(\mathbb{C}^n)$ to $\mathcal{P}_d(\mathbb{C}^n)$ for which $\Pi(p) = p$ for every $p \in \mathcal{P}_d(\mathbb{C}^n)$. Such a projector Π is said to *preserve homogeneous partial differential equations* (*HPDE*) *of degree k* if for every $f \in H(\mathbb{C}^n)$ and every homogeneous polynomial of degree k, $q(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$, we have

$$q(D)f = 0 \Rightarrow q(D)\Pi(f) = 0, \tag{1}$$

where $q(D) := \sum_{|\alpha|=k} a_\alpha D^\alpha$, $D^\alpha = \partial^{|\alpha|}/\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}$, and $|\alpha| = \sum_{j=1}^n \alpha_j$ denotes the length of the multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$.

In [3] Calvi and Filipsson give a precise description of the polynomial projectors preserving all HPDE. In particular they show that a polynomial projector preserves all HPDE as soon as it preserves HPDE of degree 1. Then naturally arises the question of the existence of polynomial projectors preserving HPDE of degree k ($k \geqslant 1$) without preserving HPDE of smaller degree. In this note we prove that such projectors do indeed exist and we extend the basic structure theorem proved in [3] to this more general case. As a consequence we show that a polynomial projector which preserves HPDE of degree k necessarily preserves HPDE of every degree not smaller than k.

We also complete the results of [3] in another direction. Calvi and Filipsson have used their results to give a new characterization of Kergin interpolation. Namely, they have shown that the interpolation space of a polynomial projector of degree d (see the definition below) that preserves HPDE contains no more than—and only Kergin interpolation projector effectively contains—$d + 1$ Dirac (point-evaluation) functionals. Here we give a characterization of Abel–Gontcharoff projectors as the only Birkhoff polynomial projectors that preserve all HPDE (the definition are recalled in the text).

In [8] (see also [9]) Petersson has settled a convenient formalism (using the concept of pairing of Banach spaces) and extended results of [3] to Banach spaces. Our Theorem 1 is likely to have a similar infinite dimensional counterpart.

The main results of this paper were announced without proof in [4].

2. Definitions and background

We recall some definitions and results from [3]. A polynomial projector Π can be completely described by the so called *space of interpolation conditions* $\Im(\Pi) \subset H'(\mathbb{C}^n)$, where $H'(\mathbb{C}^n)$ denotes the space of the linear continuous functionals on $H(\mathbb{C}^n)$ whose elements are usually called analytic functionals. The space $\Im(\Pi)$ is defined as follows : an element $\varphi \in H'(\mathbb{C}^n)$ belongs to $\Im(\Pi)$ if and only if for any $f \in H(\mathbb{C}^n)$ we have

$$\varphi(f) = \varphi(\Pi(f)).$$

Let $\{p_\alpha : |\alpha| \leqslant d\}$ be a basis of $\mathcal{P}_d(\mathbb{C}^n)$. Then we can represent Π as

$$\Pi(f) = \sum_{|\alpha| \leqslant d} a_\alpha(f) p_\alpha, \ f \in H(\mathbb{C}^n) \tag{2}$$

with some a_α*'s* $\in H'(\mathbb{C}^n)$, and $\Im(\Pi)$ is given by

$$\Im(\Pi) = \operatorname{span}\{a_\alpha : |\alpha| \leqslant d\}.$$

In particular, in (2), we may take $p_\alpha = u_\alpha$ where $u_\alpha(z) := z^\alpha/\alpha!$, $z^\alpha := \prod_{j=1}^n z_j^{\alpha_j}$, $\alpha! := \prod_{j=1}^n \alpha_j!$. Notice that the dimension of $\Im(\Pi)$ is

$$N_d(n) := \binom{n+d}{n},$$

which coincides with the dimension of $\mathcal{P}_d(\mathbb{C}^n)$. Moreover, the restriction of $\Im(\Pi)$ to $\mathcal{P}_d(\mathbb{C}^n)$ is the dual space $\mathcal{P}_d^*(\mathbb{C}^n)$.

Conversely, if $\mathbf{I}$ is a subspace of $H'(\mathbb{C}^n)$ of dimension $N_d(n)$ such that the restriction of its element to $\mathcal{P}_d(\mathbb{C}^n)$ spans $\mathcal{P}_d^*(\mathbb{C}^n)$ then there exists a unique polynomial projector $\wp(\mathbf{I})$ such that $\mathbf{I} = \Im(\wp(\mathbf{I}))$. In that case we say that $\mathbf{I}$ *is an interpolation space for* $\mathcal{P}_d(\mathbb{C}^n)$ and, for $p \in \mathcal{P}_d(\mathbb{C}^n)$, we have

$$\wp(\mathbf{I}, f) = p \Leftrightarrow \varphi(p) = \varphi(f), \quad \forall \varphi \in \mathbf{I}.$$

Notice that for every projector Π we have $\wp(\Im(\Pi)) = \Pi$.

Let Π be a polynomial projector preserving HPDE of degree 1. A function f is called *ridge function* if it is of the form $f(z) = h(a.z)$ with $h \in H(\mathbb{C})$, where

$$y \cdot z := \sum_{j=1}^n y_j \cdot z_j \ \forall y, z \in \mathbb{C}^n.$$

Using (1) with polynomials q of degree 1, we can easily see that Π also preserves ridge functions, that is, if $f(z) = h(a.z)$ then there exists a univariate polynomial p such that

$$\Pi(h(a.\cdot))(z) = p(a.z).$$

This formula defines a univariate polynomial projector which is denoted by Π_a, satisfying the following property

$$\Pi_a(h)(a.z) = \Pi(h(a.\cdot))(z).$$

Let $\mu_0, \mu_1, \ldots, \mu_d$ be $d+1$ not necessarily distinct analytic functionals on $H(\mathbb{C}^n)$ such that $\mu_i(1) \neq 0$ for $i = 0, 1, \ldots, d$. Then, it was proved in [3] that

$$\mathbf{I} := \operatorname{span}\{D^\alpha \mu_{|\alpha|} : |\alpha| \leqslant d\} \tag{3}$$

is an interpolation space for $\mathcal{P}_d(\mathbb{C}^n)$, where for an analytic functional $\varphi \in H'(\mathbb{C}^n)$ and a multi-index α, the derivative $D^\alpha \varphi$ is defined as the analytic functional given by

$$D^\alpha \varphi(f) := \varphi(D^\alpha f).$$

The projector corresponding to space **I** in (3) is called D-*Taylor projector*. It was introduced by Calvi [2].

For $\alpha \in \mathbb{Z}_+^n$ and $a \in \mathbb{C}^n$, the analytic functional $D^\alpha[a]$ is defined by

$$D^\alpha[a](f) = D^\alpha f(a), \ f \in H(\mathbb{C}^n).$$

It is called *discrete functional*. For $\alpha = 0$, we use the abbreviation: $D^0[a] = [a]$. Typical D-Taylor projector is the Abel–Gontcharoff projector when $\mu_i := [a_i]$ in (3). For other natural examples, see [1–3,7].

Theorem A (*Calvi and Filipsson [3]*). *Let* Π *be a polynomial projector of degree d in* $H(\mathbb{C}^n)$. *Then the following four conditions are equivalent.*

(1) Π *preserves all HPDE.*
(2) Π *preserves ridge functions.*
(3) Π *is a D-Taylor projector.*
(4) *There are analytic functionals* $\mu_0, \mu_1, \ldots, \mu_d \in H'(\mathbb{C}^n)$ *with* $\mu_i(1) \neq 0, \ i = 0, 1, \ldots, d$, *such that* Π *is represented in the following form*

$$\Pi(f) = \sum_{|\alpha| \leqslant d} D^\alpha \mu_{|\alpha|}(f) u_\alpha. \tag{4}$$

This theorem shows that a polynomial projector Π preserving HPDE of degree 1 also preserves all HPDE.

3. Polynomial projectors preserving HPDE

In this section we extend Theorem A (and D-Taylor representation) to polynomial projectors preserving HPDE of degree $k, \ 1 \leqslant k \leqslant d$. We recall that the Laplace transform of an analytic functional $\varphi \in H'(\mathbb{C}^n)$ is the entire function $\widehat{\varphi}$ defined by

$$\widehat{\varphi}(w) := \varphi(e_w), \ w \in \mathbb{C}^n,$$

where $e_w(z) := \exp(w.z)$. The mapping $\varphi \mapsto \widehat{\varphi}$ is an isomorphism between the space of analytic functionals and the space of entire functions of exponential type (for details, see [5, p.108]). Notice that $\widehat{[a]} = e_a$ and $(\widehat{D^\alpha \varphi})(w) = w^\alpha \widehat{\varphi}(w)$.

Theorem 1. *A polynomial projector* Π *of degree d preserves HPDE of degree* $k, \ 1 \leqslant k \leqslant d$, *if and only if there are analytic functionals* $\mu_k, \mu_{k+1}, \ldots, \mu_d \in H'(\mathbb{C}^n)$ *with* $\mu_i(1) \neq 0, \ i = k, \ldots, d$, *such that* Π *is represented in the following form*

$$\Pi(f) = \sum_{|\alpha| < k} a_\alpha(f) u_\alpha + \sum_{k \leqslant |\alpha| \leqslant d} D^\alpha \mu_{|\alpha|}(f) u_\alpha \tag{5}$$

with some a_α*'s* $\in H'(\mathbb{C}^n), \ |\alpha| < k$.

Proof. We first prove the sufficiency part of the theorem. Suppose that there are $\mu_k, \ldots, \mu_d \in H'(\mathbb{C}^n)$ with $\mu_s(1) \neq 0, \ s = k, \ldots, d$, such that Π is represented as in (5). We join any analytic functionals $\mu_0, \mu_1, \ldots, \mu_{k-1}$ with $\mu_s(1) \neq 0,\ s = 0, \ldots, k-1$, to these analytic functionals. Consider the D-Taylor projector Π' corresponding to the interpolation space $\mathrm{span}\{D^\alpha \mu_{|\alpha|} : |\alpha| \leqslant d\}$. Due to (4) Π' may be represented as

$$\Pi'(f) = \sum_{|\alpha| \leqslant d} D^\alpha \mu'_{|\alpha|}(f) u_\alpha \tag{6}$$

with $\mu'_0, \mu'_1, \ldots, \mu'_d \in H'(\mathbb{C}^n)$ and $\mu'_s(1) \neq 0,\ s = 0, 1, \ldots, d$. From the last representation and (5), we derive that

$$D^\alpha \mu_{|\alpha|} = \sum_{|\beta| \leqslant d} c_\beta D^\beta \mu'_{|\beta|}, \quad k \leqslant |\alpha| \leqslant d. \tag{7}$$

Applying both sides of (7) to $u_\beta,\ |\beta| \leqslant d$, we get

$$D^\alpha \mu_{|\alpha|} = D^\alpha \mu'_{|\alpha|}, \quad k \leqslant |\alpha| \leqslant d.$$

Hence,

$$\Pi'(f) = \sum_{|\alpha|<k} D^\alpha \mu'_{|\alpha|}(f) u_\alpha + \sum_{k \leqslant |\alpha| \leqslant d} D^\alpha \mu_{|\alpha|}(f) u_\alpha.$$

We now prove that Π preserves HPDE of degree k. Let q be a homogeneous polynomial of degree k and $f \in H(\mathbb{C}^n)$ such that $q(D)f = 0$. We prove that $q(D)\Pi(f) = 0$. Indeed, since

$$q(D)\Big(\sum_{|\alpha|<k} D^\alpha \mu'_{|\alpha|}(f) u_\alpha\Big) = q(D)\Big(\sum_{|\alpha|<k} a_\alpha(f) u_\alpha\Big) = 0,$$

we have

$$q(D)(\Pi(f)) = q(D)(\Pi'(f)) = q(D)\Big(\sum_{k \leqslant |\alpha| \leqslant d} D^\alpha \mu_{|\alpha|}(f) u_\alpha\Big).$$

Because the D-Taylor projector Π' preserves HPDE of degree k, we obtain that

$$q(D)(\Pi(f)) = q(D)(\Pi'(f)) = 0.$$

We pass to the necessary part of the theorem. Consider the following representation of Π

$$\Pi(f) = \sum_{|\alpha| \leqslant d} a_\alpha(f) u_\alpha.$$

Take a point $w \in \mathbb{C}^n$ with $w \neq 0$. Suppose that $(c_s)_{|s|=k},\ c_s \in \mathbb{C}$, is a sequence such that

$$\sum_{|s|=k} c_s w^s = 0. \tag{8}$$

For the homogeneous polynomial of degree k

$$q(z) := \sum_{|\alpha|=k} c_s z^s,$$

by (8) we have

$$q(D)(e_w) = 0.$$

Since Π preserves HPDE of degree k, we obtain that

$$q(D)(\Pi(e_w)) = 0.$$

From the identity

$$\Pi(e_w) = \sum_{0 \leqslant |\alpha| \leqslant d} \widehat{a}_\alpha(w) u_\alpha,$$

we derive that

$$F(z) := \sum_{|s|=k} \sum_{|\alpha| \leqslant d} c_s \widehat{a}_\alpha(w) \Big(D^s u_\alpha(z) \Big) = 0. \tag{9}$$

By virtue of the equality

$$D^s(u_\alpha) = \begin{cases} u_{\alpha-s}, & \alpha \geqslant s, \\ 0, & \text{otherwise}, \end{cases}$$

we have

$$\begin{aligned} F(z) &= \sum_{|s|=k} c_s \sum_{|\alpha| \leqslant d,\ \alpha \geqslant s} \widehat{a}_\alpha(w) u_{\alpha-s}(z) \\ &= \sum_{|s|=k} c_s \sum_{|\beta| \leqslant d-k} \widehat{a}_{\beta+s}(w) u_\beta(z) \\ &= \sum_{|\beta| \leqslant d-k} \Big(\sum_{|s|=k} c_s \widehat{a}_{\beta+s}(w) \Big) u_\beta(z). \end{aligned}$$

This means that F is a polynomial of degree $d - k$ that is identically equal to zero due to (9). Hence, we proved that if for $(c_s)_{|s|=k},\ c_s \in \mathbb{C}$, there holds (8), then we have

$$\sum_{|s|=k} c_s \widehat{a}_{\beta+s}(w) = 0, \quad |\beta| \leqslant d - k. \tag{10}$$

We will prove that if β and w are fixed so that $w \neq 0$ and $|\beta| \leqslant d - k$, then

$$\frac{\widehat{a}_{\beta+s}(w)}{w^{\beta+s}} = \text{const} \tag{11}$$

for every s with $|s| = k$ (for convenience we put $\frac{0}{0} = 0$). There are two cases:

Case A: $w^\beta \neq 0$. In this case, for $(c_s)_{|s|=k},\ c_s \in \mathbb{C}$, the equality

$$\sum_{|s|=k} c_s w^{\beta+s} = 0 \tag{12}$$

implies that

$$\sum_{|s|=k} c_s w^s = 0. \tag{13}$$

By (10) we deduce the implication

$$\sum_{|s|=k} c_s w^s = 0 \Rightarrow \sum_{|s|=k} c_s \widehat{a}_{\beta+s}(w) = 0. \tag{14}$$

Notice that if $a, b \in \mathbb{C}^m,\ b \neq 0$ are given and the equality $b.c = 0$ implies $a.c = 0$ for $c \in \mathbb{C}^m$, then

$$\frac{a_j}{b_j} = \text{const},\ \ j = 1, \ldots, m.$$

Therefore, from (12), (13) and (14) we prove (11).

Case B: $w^\beta = 0$. In this case, we will show that

$$\widehat{a}_{\beta+s}(w) = 0,\ \ |s| = k. \tag{15}$$

Fix s^0 with $|s^0| = k$. Since $w^\beta = 0$ we can rewrite $s^0 + \beta = s^1 + \beta^1$ so that $|s^1| = k,\ |\beta^1| = |\beta| \leqslant d - k$, and $w^{s^1} = 0$. Applying (10) to w and β^1 gives the implication

$$\sum_{|s|=k} c_s w^s = 0 \Rightarrow \sum_{|s|=k} c_s \widehat{a}_{s+\beta^1}(w) = 0.$$

Hence, we have

$$\frac{\widehat{a}_{s+\beta^1}(w)}{w^s} = \text{const},\ \ |s| = k.$$

In particular, for $s = s^1$

$$\widehat{a}_{s^0+\beta}(w) = \widehat{a}_{s^1+\beta^1}(w) = 0.$$

Thus, (15) has been proved. Further, we will prove that if α^1 and α^2 are multi-indices with length $i,\ k \leqslant i \leqslant d$, then

$$\frac{\widehat{a}_{\alpha^1}(w)}{w^{\alpha^1}} = \frac{\widehat{a}_{\alpha^2}(w)}{w^{\alpha^2}}. \tag{16}$$

The special case when $\alpha^1 = s^1 + \beta$ and $\alpha^2 = s^2 + \beta$ for some β with $|\beta| \leqslant d - k$, follows from (11). This case also implies that if α is a multi-index with length $i,\ k \leqslant i \leqslant d$, and $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}^n$ is a vector such that

$$\sum_{j=1}^n |\delta_j| \leqslant k, \quad \sum_{j=1}^n \delta_j = 0, \quad \alpha + \delta \in \mathbb{Z}^n_+,$$

then

$$\frac{\widehat{a}_{\alpha+\delta}(w)}{w^{\alpha+\delta}} = \frac{\widehat{a}_{\alpha}(w)}{w^{\alpha}}. \tag{17}$$

Let us now prove the general case of (16). Obviously, the case $i = k$ follows from (11) with $\beta = 0$. Consider the case $i > k$. Notice that if α^1 and α^2 are multi-indices with length i, then there are $\delta^1, \ldots, \delta^l \in \mathbb{Z}^n$, $(l \leqslant i)$ such that

$$\sum_{j=1}^{n} |\delta_j^s| \leqslant k, \quad \sum_{j=1}^{n} \delta_j^s = 0, \quad \alpha^1 + \sum_{s=1}^{m} \delta^s \in \mathbb{Z}_+^n, \quad m = 1, \ldots, l,$$

and

$$\alpha^2 = \alpha^1 + \sum_{s=1}^{l} \delta^s.$$

Applying (17) l times gives (16). By virtue of (16), we can write

$$\widehat{a}_{\alpha}(w) = b_i(w) w^{\alpha}, \quad w \neq 0, \quad |\alpha| = i, \quad k \leqslant i \leqslant d. \tag{18}$$

Since $\widehat{a}_{\alpha}$ is an entire function of exponential type, taking $\alpha = \alpha(i) = (i\delta_{ij})_{j=1,\ldots,n}$, we conclude that b_i extends to an holomorphic function on $\mathbb{C}_s^n := \mathbb{C}^n \setminus \{w : w_s = 0\}$, and consequently on $\bigcup_{s=1}^{n} \mathbb{C}_s^n = \mathbb{C}^n \setminus \{0\}$. Because an holomorphic function of several complex variables has no isolated singularities (see [10, Ch.III, §11]), we may deduce that each b_i extends uniquely to an entire function which is again denoted by b_i. Moreover, by (18) b_i must be an entire function of exponential type too and, therefore, the Laplace transform of an analytic functional μ_i, e.g., $b_i(w) = \widehat{\mu}_i(w)$. Thus we have

$$\widehat{a}_{\alpha}(w) = w^{\alpha} \widehat{\mu}_i(w).$$

By the identity

$$(\widehat{D^{\alpha}\mu_i})(w) = w^{\alpha} \widehat{\mu}_i(w)$$

we obtain

$$a_{\alpha} = D^{\alpha} \mu_i, \quad i = k, \ldots, d.$$

Summing up we arrive at the conditions that there are analytic functionals $\mu_k, \mu_{k+1}, \ldots, \mu_d \in H'(\mathbb{C}^n)$ such that Π is represented as in (5) with some a_{α}'s $\in H'(\mathbb{C}^n)$, $|\alpha| \leqslant k$. $\square$

From Theorem 1 we can derive some interesting properties of polynomial projectors preserving HPDE of degree k. First of all, observe that the equivalence of Conditions 1 and 4 in Theorem A immediately follows from Theorem 1.

Corollary 1. *If the polynomial projector* Π *of degree* d *preserves HPDE of degree* k, $1 \leqslant k \leqslant d$, then Π *preserves also HPDE of all degree greater than* k.

Corollary 2. *If $1 < k \leqslant d$, there is a polynomial projector of degree d which preserves HPDE of degree k but not HPDE of all degree smaller than k.*

Proof. Observe that as the set $\{u_\alpha : |\alpha| \leqslant d\}$ is linearly independent, there exist distinct $\mu_1, \mu_2 \in H'(\mathbb{C}^n)$ such that

(i) $\mu_j(1) = 1,\ j = 1, 2$
(ii) $\mu_j(u_\alpha) = 0,\ 1 \leqslant |\alpha| \leqslant d,\ j = 1, 2.$

Fix two multi-indices α^1, α^2 with $|\alpha^1| = |\alpha^2| = k - 1$. We have

$$D^{\alpha^j}\mu_j(u_\beta) = \delta_{\alpha^j\beta}, \quad j = 1, 2.$$

Consider the polynomial projector Π of degree d defined by

$$\Pi(f) = \sum_{j=1}^{2} D^{\alpha^j}\mu_j(f)u_{\alpha^j} + \sum_{|\alpha| \leqslant d,\ \alpha \neq \alpha^1, \alpha^2} D^\alpha[0](f)u_\alpha. \tag{19}$$

Observe that Π is of the form (5), and by Theorem 1 Π preserves HPDE of degree k. Suppose now that Π also preserves HPDE of degree $k - 1$. Again by Theorem 1 Π may be represented as follows

$$\Pi(f) = \sum_{|\alpha| \leqslant k-2} a_\alpha(f)u_\alpha + \sum_{k-1 \leqslant |\alpha| \leqslant d} D^\alpha\mu_{|\alpha|}(f)u_\alpha. \tag{20}$$

Comparing (19) and (20) gives

$$D^{\alpha^j}\mu_j = D^{\alpha^j}\mu_{k-1}, \quad j = 1, 2.$$

Passing to the Laplace transform we have

$$w^{\alpha^j}\widehat{\mu}_j(w) = w^{\alpha^j}\widehat{\mu}_{k-1}(w), \quad j = 1, 2.$$

Hence, by the uniqueness principle we can easily see that

$$\widehat{\mu}_1 = \widehat{\mu}_2 = \widehat{\mu}_{k-1}$$

because $\widehat{\mu}_i$ and $\widehat{\mu}_{k-1}$ are entire functions. Consequently, $\mu_1 = \mu_2$. This contradicts our construction of μ_1 and μ_2. Thus it has been proved that Π does not preserve HPDE of degree $k - 1$. By use of Corollary 1 we deduce that Π does not also preserve HPDE of any degree smaller than k. $\square$

Corollary 3. *Let Π be a polynomial projector of degree d preserving HPDE of degree $k,\ 1 \leqslant k \leqslant d$. Then there are functionals $\mu_k, \mu_{k+1}, \ldots, \mu_d$ with $\mu_i(1) = 1$ $(k \leqslant i \leqslant d)$ such that the set*

$$\text{span}\{D^\alpha\mu_s : |\alpha| = s,\ s = k, \ldots, d\}$$

is a proper subset of $\Im(\Pi)$. *Moreover, if* Π *is represented as in* (5) *with* $\mu_i(1) = 1,\ i = k, \ldots, d$, *and if, for some* β *with* $|\beta| \geqslant k$, *we have* $D^\beta v \in \Im(\Pi)$ *then there exists a relation*

$$v = \mu_{|\beta|} + \sum_{j=1}^{d-|\beta|} \sum_{l_1 l_2 \ldots l_j} c_{l_1 l_2 \ldots l_j} \frac{\partial^j \mu_{|\beta|+j}}{\partial z_{l_1} \ldots \partial z_{l_j}}, \tag{21}$$

where each l_k *is taken over* $\{1, 2, \ldots, n\}$.

Proof. We just need to prove (21). It follows from (5) that the map

$$f \mapsto \sum_{|\alpha| \leqslant k-1} a_\alpha(f) u_\alpha$$

is a polynomial projector of degree $k-1$. In particular, the restrictions of the a_α's are linearly independent on $\mathcal{P}_{k-1}(\mathbb{C}^n)$. Now, for some coefficients c_α we have

$$D^\beta v = \sum_{|\alpha| \leqslant k-1} c_\alpha a_\alpha + \sum_{k \leqslant |\alpha| \leqslant d} c_\alpha D^\alpha \mu_{|\alpha|}. \tag{22}$$

If follows that for every γ with $|\gamma| \leqslant k-1$, we have

$$0 = \sum_{|\alpha| \leqslant k-1} c_\alpha a_\alpha(u_\gamma).$$

Since the restrictions of the a_α's are linearly independent on $\mathcal{P}_{k-1}(\mathbb{C}^n)$ we deduce that $c_\alpha = 0$ for $|\alpha| \leqslant k-1$. Hence we have

$$D^\beta v = \sum_{k \leqslant |\alpha| \leqslant d} c_\alpha D^\alpha \mu_{|\alpha|}.$$

Now taking the Laplace transforms of both sides and expanding them in power series, we find by identifying the coefficients that every coefficient c_α must vanish if α does not belong to $Z(\beta)$ which is the set of all multi-indices α such that $|\alpha| \leqslant d,\ \alpha_j \geqslant \beta_j,\ j = 1, \ldots, n,\ \alpha \neq \beta$. Taking out the common factor w^β on both sides and returning to the functionals, we obtain the claimed representation. $\square$

4. A characterization of Abel–Gontcharoff projectors

A *Birkhoff projector* is called a polynomial projector Π for which $\Im(\Pi)$ is generated by discrete functionals that is to say by functionals of the form $D^\alpha[a]$. For results on Birkhoff interpolation we refer to [6]. The following theorem might seem intuitively clear but we found no immediate proof. It is worth noting that this result is typical for the higher dimension. It is indeed not true in dimension 1 in which the concept of projector preserving all HPDE reduces to a triviality.

Theorem 2. *Let Π be a Birkhoff projector of degree d on $\mathbb{C}^n$, $n \geqslant 2$. Then Π preserves all HPDE if and only if it is an Abel–Gontcharoff projector, that is, there are $a_0, \ldots, a_d \in \mathbb{C}^n$ not necessary distinct such that*

$$\Im(\Pi) = \mathrm{span}\{\, D^\alpha[a_s] : |\alpha| = s,\ s = 0, \ldots, d\,\}.$$

Proof. The sufficiency part is trivial. We prove that the condition is necessary. For $k = 0, 1, \ldots, d$, we set

$$A^k(\Pi) := \{a \in \mathbb{C}^n : \exists\alpha,\ |\alpha| = k,\ D^\alpha[a] \in \Im(\Pi)\}$$

and $s_k = \mathrm{card} A^k(\Pi)$. When $s_k \neq 0$ we define

$$A^k(\Pi) = \{a_k^1, a_k^2, \ldots, a_k^{s_k}\},$$

where $a_k^{j'} \neq a_k^j$ for $j' \neq j$. It has been shown in [3, Proposition 2] that if Π is a polynomial projector preserving all HPDE and if $D^\alpha[a] \in \Im(\Pi)$ for one multi-index α of length k, then $D^\beta[a] \in \Im(\Pi)$ for every multi-index β of the same length k. This implies that

$$\{D^\alpha[a_k^j] : k \in J,\ |\alpha| = k,\ j = 1, \ldots, s_k\} \subset \Im(\Pi),$$

where J is the set of all k such that $A^k(\Pi)$ is not empty. The set on the left-hand side above will be denoted by $\Gamma(\Pi)$. Since this set is linearly independent and since Π is a Birkhoff projector it forms a basis of $\Im(\Pi)$. Thus, what we need to prove is only that $s_k = 1$ for $k = 0, 1, \ldots, d$. For $b \in \mathbb{C}^n$, $b \neq 0$, we define the linear mapping $b \star \cdot$ from $H'(\mathbb{C}^n)$ into $H'(\mathbb{C})$ as follows

$$(b \star \varphi)(h) := \varphi(h(b.\cdot)).$$

It was proved in [3, Proposition 1] that the restriction of this mapping to $\Im(\Pi)$ is a linear mapping from $\Im(\Pi)$ onto $\Im(\Pi_b)$ (see Section 2 for the notation), that is,

$$b \star \Im(\Pi) = \Im(\Pi_b).$$

Step 1: We prove that $\sum_{k=0}^d s_k = d + 1$. As $n > 1$, we can choose a (nonzero) $b \in \mathbb{C}^n$ so that for $k \in J$ the elements $b.a_k^j$, $j = 1, \ldots, s_k$, are pairwise distinct. From the equality

$$b \star D^\alpha[a_k^j] = b^\alpha D^k[b.a_k^j]$$

for any multi-index α of length k, and the existence of a multi-index α of length k such that $b^\alpha \neq 0$, we deduce that the discrete functionals

$$D^k[b.a_k^j], \quad k \in J, \quad j = 1, \ldots, s_k, \tag{23}$$

span $\Im(\Pi_b)$. Moreover, because the s_k complex numbers $b.a_k^j$ are pairwise distinct, the discrete functionals in (23) are linearly independent and, therefore, form a basis of $\Im(\Pi_b)$. Consequently, we have

$$\dim(\Im(\Pi_b)) = \sum_{k \in J} s_k \tag{24}$$

and the claim is proved since $\dim(\Im(\Pi_b)) = d + 1$ and $\sum_{k \in J} s_k = \sum_{k=0}^{d} s_k$.

Step 2: We prove that $s_k \leqslant 1$ for $k = 0, \ldots, d$. Suppose to the contrary that there is k_0 such that $s_{k_0} \geqslant 2$. We can choose a nonzero $b^* \in \mathbb{C}^n$ so that $b^*.a^1_{k_0} = b^*.a^2_{k_0}$. Then the set

$$\{D^{k_0}[b^*.a^j_{k_0}],\ j = 1, \ldots, s_{k_0}\}$$

contains at most $s_{k_0} - 1$ functionals. Consequently, the cardinality of the set

$$\{D^k[b^*.a^j_k], k \in J,\ j = 1, \ldots, s_k\} \tag{25}$$

is at most $(\sum_{k\in J} s_k) - 1$ which is, in view of Step 1, not greater than d. This is a contradiction, because the functionals in (25) span $\Im(\Pi_{b^*})$ which has dimension $d + 1$.

The two steps imply that $s_k = 1$ for $k = 0, \ldots, d$, and this concludes the proof. $\square$

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